Response of massive bodies to gravitational waves

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Abstract

The response of a massive body to gravitational waves is described on the microscopic level, taking the metric perturbation of the electromagnetic and gravitational forces into account. The effects found substantially differ from those obtained in the commonly used oscillator model. The electromagnetic coupling induces a dominant surface effect, the gravitational coupling gives rise to the excitation of quadrupole modes, but several orders of magnitude smaller.

1 Introduction

Gravitational waves were already considered by Einstein as the wave solutions of the linearized field equations of gravity. There is indirect evidence of their existence through systems of binary pulsars that loose energy in form of gravitational radiation [1], their direct experimental measurement presently is one of the most challenging tasks in gravitational physics. Very sensitive detectors operating at the quantum limit are needed to detect directly gravitational waves from cosmic events such as collapsing or colliding star systems. There are basically two different types of detectors: resonant mass antennas based on the resonant excitation of quadrupole-type modes of a appropriately chosen massive body, like the bar detectors conceived by Weber [2], and laser interferometric devices that detect the direction-dependent variation of the proper distance between the mirrors of a Michelson interferometer [3]. Detectors of both types are presently under construction [4].

Commonly a resonant mass antenna is described in Riemannian normal coordinates with respect to its center of mass, the proper frame of reference (PFR). In this frame the energy input is calculated, the detector typically is idealized by a spring that couples two masses [2]. It was our intention to validate the results obtained from this simple model by microscopic considerations. We were surprised to find a rather different result, namely that the effect of gravitational waves on a massive body is restricted to the surface. We are convinced that our results are correct, and that there is a decisive error is the common line of argumentation. The common argument is [5] that in the PFR the metric of the gravitational wave field leads to variations of the electromagnetic field of order

$$\delta A/A \sim \left(L^2/\lambda^2\right) h^{TT} \tag{1}$$

where L is the distance from the origin, λ and h^{TT} the wave length and amplitude of the gravitational wave, respectively, whereas the mechanical forces lead to displacements of the constituent particles of the body that in turn lead to variations of the electromagnetic field of order

$$\delta A/A \sim h^{TT} \tag{2}$$

so that the metric effect can be neglected. Based on this argument, the electromagnetic field is considered to be not affected by the gravitational wave in the PFR. This argument should be revisited, for the following reason. The mechanical forces on the masses are induced by the metric component g_{00} that acts as a potential. In the PFR g_{00} is given by

$$g_{00} = R_{0i0l} x^i x^l \sim \ddot{h}^{TT} L^2 \sim \left(L^2/\lambda^2\right) h^{TT}. \tag{3}$$

Thus the gravitational wave induces changes of the mechanical potential that are of the same order as those of the electromagnetic potential (1). Hence both effects have to be taken into account. If the electromagnetic effects are ignored the mechanical forces lead to resonant amplitudes of order h^{TT}. Comparing these amplitudes that give rise to (2) with the changes in the potential (1) is inconsistent. We will show in this article, that the electromagnetic and mechanical forces cancel in the bulk of the material, so that basically no resonance of the quadrupole modes will occur.

2 Frames of reference

The use of the PFR is quite natural, but it is not so well suited for the study of the detector on the microscopic level, for two reasons. First, in such a coordinate system the interaction of the constituent particles will depend not only on their mutual distance, but also on their positions relative to the center of mass. Second, energy-momentum conservation, one of our main tools, is easier to handle in the reference system of the plane wave, conventionally chosen with transverse-traceless (TT) gauge [2]. Therefore we have chosen the TT system for our considerations.

We first show that in the TT system no energy is transferred to a system of non-interacting particles, then extend the discussion to electromagnetically coupled systems, and see how an effective transfer of energy and momentum is achieved. Here we should point out that conserved energy and momentum are not covariant quantities and depend on the frame of reference [6]; as is well known the notion of local energy densities is not well defined in general relativity. The corresponding quantities introduced are helpful for the calculations, but do not correspond to measurable quantities. Therefore we must take care to compare only well defined quantities with experiment. This can be achieved e.g. by comparing the relevant quantities before and after a wave pulse or wave train, when all frames of reference coincide with the Minkowski frame in which we can interpret the result.

3 Particle Motion

A point-like test mass with electric charge e can be described in the presence of electromagnetic fields and gravitation by the Hamiltonian [7]

$$H = c\sqrt{(p_\mu - eA_\mu)\, g^{\mu\nu}\, (p_\nu - eA_\nu)} \tag{4}$$

where we use coordinates $x^\mu, \mu = 0, 1, 2, 3, 4$ for space-time with metric $g_{\mu\nu}(x^\kappa)$ and signature $+ - --$, p_μ are the momentum coordinates in the cotangent space, and $A_\mu(x^\kappa)$ is the electromagnetic four-potential. We also use the notation (ct, x, y, z) in an obvious manner. The evolution parameter will be denoted by τ. The Hamiltonian is conserved, $\partial H/\partial\tau = 0$, it represents the rest mass $m = H/c^2$of the particle. The canonical equations of motion are given by

$$\dot{x}^\mu = \frac{\partial H}{\partial p_\mu}, \qquad \dot{p}_\mu = -\frac{\partial H}{\partial x^\mu}. \tag{5}$$

The constancy of H is equivalent to $\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu = c^2$ for any trajectory, thus the evolution parameter is the proper time.

A gravitational wave propagating in z-direction with $+$ and $\times$ polarization modes is described in TT gauge by the metric tensor

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 + f_+(ct-z) & f_\times(ct-z) & 0 \\ 0 & f_\times(ct-z) & -1 - f_+(ct-z) & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \eta_{\mu\nu} + h_{\mu\nu} \tag{6}$$

where $h_{\mu\nu}$ is a small perturbation of the Minkowski metric $\eta_{\mu\nu}$ [2]. In the absence of an electromagnetic field we obtain for this metric a Hamiltonian that leads to four conserved quantities: H, p_x, p_y, and $p_0 + p_3 = E/c + p_z$ where E is the energy of the particle. Note that $p_3 = -m\dot{z}$, so that the difference between the energy and the conventional z-momentum is conserved. This is natural, since the gravitational wave not only carries energy E_w, but also momentum P_w with the relation $E_w = c\,|P_w|$ that holds for all massless objects in special relativity. The exchange ΔE of energy between the wave and a test mass thus is always accompanied with an exchange ΔP of momentum:

$$\Delta E = c\Delta P. \tag{7}$$

The existence of four conserved quantities now allows us to integrate the equations of motion completely:

$$\begin{array}{ll} \dot{x}^0 = \frac{1}{m} p_0 & \dot{x}^2 = \frac{1}{m}\left(g^{21}p_1 + g^{22}p_2\right) \\ \dot{x}^1 = \frac{1}{m}\left(g^{11}p_1 + g^{12}p_2\right) & \dot{x}^3 = -\frac{1}{m}p_3 \\ p_1 = const & p_2 = const \end{array} \tag{8}$$

$$\begin{aligned} p_0 &= \frac{1}{2}\left(p_0 + p_3\right) + \frac{1}{2}\frac{m^2}{\left(p_0 + p_3\right)}\left(c^2 - \frac{1}{m^2}p_a g^{ab} p_b\right) \\ p_3 &= \frac{1}{2}\left(p_0 + p_3\right) - \frac{1}{2}\frac{m^2}{\left(p_0 + p_3\right)}\left(c^2 - \frac{1}{m^2}p_a g^{ab} p_b\right) \end{aligned} \tag{9}$$

where we use the indices a, b for a summation over $1, 2$ only. Let us consider a wave pulse. We denote the initial conditions before the arrival of the pulse by $\bar{p}_\mu$, and have

$$\begin{aligned} p_0(\tau) &= \bar{p}_0 - \frac{1}{2}\frac{p_a h^{ab} p_b}{\bar{p}_0 + \bar{p}_3} \\ p_3(\tau) &= \bar{p}_3 + \frac{1}{2}\frac{p_a h^{ab} p_b}{\bar{p}_0 + \bar{p}_3} \end{aligned} \tag{10}$$

where $h^{ab} = g^{ab} - \eta^{ab}$. Thus after the pulse, where the perturbation h is zero again, the particle has the same four-momentum as before, and the only possible effect is a displacement of the straight trajectory after the pulse from the one before the pulse. If a particle is initially at rest, it stays at rest in this reference frame. Two particles that are at rest relative to each other, remain at rest relative to each other, though the proper distance between them changes with the wave amplitude. Thus free test particles do not take up energy from a gravitational wave. We have to consider the coupling between particles, which is basically of electromagnetic nature in a massive body, in order to describe the response of a detector. Because in the ground state of a body the constituent atoms are at rest relative to each other except for quantum effects we have no mechanical stress induced by the gravitational wave. Rotational motion with respect to the TT frame of reference must be taken into account. Internal motion, thermal or otherwise, of the atoms leads to forces of order $m\Delta v \partial h/\partial t$; we do not consider this kind of phonon-graviton interaction in this article, because it is the idea of the Weber detector to excite the quadrupole modes, not to enhance already excited modes.

4 Electromagnetic Field

We first look at the Coulomb potential generated by a charge that resides in the field of the wave. We assume that the wave field is slowly varying and the velocity of the charge relative to the source of the fields is so small that magnetic fields can be ignored. The corresponding equation to solve for the electromagnetic potential generated by a charge q_s at rest at $x = y = z = 0$ is

$$\partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu A^0 = \frac{q_s}{\varepsilon_0} \delta^3(x, y, z). \tag{11}$$

A solution correct to first order in the perturbation is simply given by

$$A^0_s(\vec{r}, ct - z) = \frac{q_s}{4\pi\varepsilon_0 r}, \quad \vec{r} = \left(x^i\right)_{i=1,2,3}, \quad r^2 = -x^i g_{ij}\left(ct - z\right) x^j \quad . \tag{12}$$

The Euclidean distance from the source is replaced by the metric distance, the electromagnetic field follows the metric changes instantaneously. In the following we assume that this principle may also by applied to other, phenomenological potentials, because the time scale set by the gravitational waves is by far larger than that of the induced changes of all other fundamental interactions. The Coulomb force between two charged particles will oscillate in phase with the gravitational wave. It is not hard to see that the local energy density of the electromagnetic field, though time-dependent, is only relocated, so that the integrated energy density does not change to first order in h, implying that radiation effects are at most of second order in h.

5 Many particles

We make a potential approximation for the many-particle Hamiltonian, since in a fully relativistic approach we had to include necessarily the dynamics of the electromagnetic field in order to preserve energy-momentum conservation. Thus we write the total Hamiltonian for many particles with coordinates $\left(x^{(s)\mu}, p_\mu^{(s)}\right)$, $s = 1, 2, ...$ as

$$H = \sum_s H^{(s)} \tag{13}$$

where

$$H^{(s)} = -\frac{1}{2m_1} p_i^{(s)} g^{ij}\left(ct - z^{(s)}\right) p_j^{(s)} + \sum_{s\prime \neq s} \frac{1}{2} V_{ss\prime}(\vec{r}^{(s)} - \vec{r}^{(s\prime)}, ct - z^{(s\prime)}) \tag{14}$$

is the contribution from a single particle. $V_{ss\prime}$ is the total potential generated by the particle s', acting on particle s. As it should be, to each particle only half the potential energy is attributed. In the electromagnetic case, the other half, as well as the infinite self-energy is subtracted with the contribution from the electromagnetic field energy [6]. We assume that the potential is a central potential depending only on the metric distance:

$$V_{ss\prime}(\vec{r}^{(s)} - \vec{r}^{(s\prime)}, ct - z^{(s\prime)}) = V^0_{ss\prime}\left(\sqrt{-\left(\vec{r}^{(s)} - \vec{r}^{(s\prime)}\right)^i g_{ij}\left(ct - z^{(s\prime)}\right)\left(\vec{r}^{(s)} - \vec{r}^{(s\prime)}\right)^j}\right),$$

as derived for the Coulomb potential.

The evolution parameter is the time t, common to all particles. The Hamiltonian (13) conserves the difference between the total energy and the center of mass z-momentum:

$$\frac{d}{dt}\left(H + c\sum_s p_3^{(s)}\right) = 0. \tag{15}$$

But since we have less conservation laws than coordinates, energy and momentum transfer from the wave to the particle system has become possible. The change of the total energy is calculated using the equations of motion

$$\begin{aligned} \frac{d}{dt}p_z^{(s)} &= -\frac{\partial H}{\partial z^{(s)}} \\ &= -\frac{1}{2m_s} p_i^{(s)} h^{ij\prime}\left(ct - z^{(s)}\right) p_j^{(s)} \\ &\quad -\frac{1}{2}\frac{\partial}{\partial z^{(s)}}\left(\sum_{s\prime \neq s} V_{s\prime s}(\vec{r}^{(s)} - \vec{r}^{(s\prime)}, ct - z^{(s)}) + \sum_{s' \neq s} V_{ss\prime}(\vec{r}^{(s\prime)} - \vec{r}^{(s)}, ct - z^{(s\prime)})\right) \end{aligned} \tag{16}$$

that lead us to

$$\begin{aligned} \frac{d}{dt}\sum_s p_z^{(s)} &= -\sum_s \frac{1}{2m_s} p_a^{(s)} h^{ab\prime}\left(ct - z^{(s)}\right) p_b^{(s)} \\ &\quad + \frac{1}{2}\sum_{s, s' \neq s} \partial_2 V_{s\prime s}(\vec{r}^{(s)} - \vec{r}^{(s')}, ct - z^{(s)}) \end{aligned} \tag{17}$$

where $h^{ij\prime}$ denotes the derivative, $\partial_2 V_{s\prime s}$ the partial derivative with respect to the second argument only. The derivatives of $V_{s\prime s}$ with respect to the first argument cancel in the sum because of the dependence on $\vec{r}^{(s)} - \vec{r}^{(s')}$ only. To first order in the perturbation we can approximate

$$\partial_2 V_{s\prime s}(\vec{r}^{(s)} - \vec{r}^{(s')}, ct - z^{(s)}) \simeq -V^{0\prime}_{s\prime s}\left(r_{ss\prime}\right)\frac{x^a_{ss\prime} x^b_{ss\prime}}{2r_{ss\prime}} h'_{ab}\left(ct - z^{(s)}\right) \tag{18}$$

where we use $x^a_{ss\prime} = \left(\vec{r}^{(s)} - \vec{r}^{(s\prime)}\right)^a$ for short, and $r_{ss\prime}$ is the unperturbed Euclidean distance. Thus we have (with $h^{ab} = -h_{ab}$ in lowest order)

$$\frac{d}{dt}\sum_s p_z^{(s)} = \sum_s h'_{ab}\left(ct - z^{(s)}\right)\left[\frac{1}{2m_s} p_a^{(s)} p_b^{(s)} - \sum_{s' \neq s} V^{0\prime}_{s\prime s}\left(r_{ss\prime}\right)\frac{x^a_{ss\prime} x^b_{ss\prime}}{2r_{ss\prime}}\right]. \tag{19}$$

For a pulse of duration T that interacts with the particle system the change of the center of mass z-momentum is given by

$$\Delta P_z = \sum_s \int_{t_0}^{t_0+T} h'_{ab}\left(ct - z^{(s)}\right) q_{ab}^{(s)}(t)dt \tag{20}$$

where $q_{ab}^{(s)}$ is the microscopic contribution from a single particle:

$$q_{ab}^{(s)}(t) = \frac{1}{2m_s} p_a^{(s)} p_b^{(s)} - \sum_{s' \neq s} V_{s's}^{0\prime}(r_{ss'}) \frac{x_{ss'}^a x_{ss'}^b}{2r_{ss'}}. \tag{21}$$

This contribution is quadrupole-like, but it is weighted with derivative of the potential. Then the sum or the integral over all particles leads in the case of the Coulomb potential to alternating sums where contributions cancel, similar to the summation leading to the Madelung constant. On the other hand, if the potential is purely attractive, as for gravitation, this effect does not occur, giving rise to a completely different response, as well will see.

For a small detector we can assume that the wave field is constant over the particle system represented by the center-of-mass coordinate $z(t)$, so we can change the integration variable to $\tau = t - z(t)/c$ and obtain

$$\Delta P_z = \int_{-\infty}^{+\infty} d\tau h'_{ab}(c\tau) \sum_s \frac{q_{ab}^{(s)}(t(\tau))}{1 - \dot{z}(t(\tau))/c} = \int_{-\infty}^{+\infty} d\tau h'_{ab}(c\tau) Q_{ab}(\tau) \tag{22}$$

with

$$Q_{ab}(\tau) = \sum_s \frac{q_{ab}^{(s)}(t(\tau))}{1 - \dot{z}(t(\tau))/c}. \tag{23}$$

For a wave pulse or wave train with duration T, such that $h_{ab}(0) = h_{ab}(cT) = 0$ we use partial integration to write (22) as

$$\Delta E = c\Delta P_z = -\int_{-\infty}^{+\infty} d\tau h_{ab}(c\tau) Q'_{ab}(\tau) \tag{24}$$

Thus the response of the detector to a gravitational wave pulse travelling in z-direction is described by the time-dependence of the microscopic function Q_{ab}. Alternatively, we may use Fourier transform to arrive at

$$\Delta E = \int_{-\infty}^{+\infty} d\nu \hat{h}_{ab}^*(\nu) i\nu \hat{Q}_{ab}(\nu) \tag{25}$$

with the spectral decompositions of the wave:

$$\hat{h}_{ab}(\nu) = \int h_{ab}(c\tau) e^{-2\pi i\nu\tau} d\tau, \tag{26}$$

and the particle system:

$$\hat{Q}_{ab}(\nu) = \sum_s \int \frac{q_{ab}^{(s)}(t(\tau))}{1 - \dot{z}(t(\tau))/c} e^{-2\pi i\nu\tau} d\tau. \tag{27}$$

Thus $\hat{Q}_{ab}$ represents the effective cross section of the particle system on the microscopic level.

6 Detector Types

From the structure of the microscopic quantity $Q_{ab}(\tau)$ we can immediately identify the different types of detectors. If the body is rotating relative to the TT frame with some frequency ω, then the momentum contribution

$$\sum_s \frac{1}{2m_s} p_a^{(s)} p_b^{(s)} \sim \omega^2 \cos 2\omega t \tag{28}$$

dominates and gives rise to a response of order h. This type of detector was first suggested by Braginsky [8, 2]. We estimate the response of a rotating mass to the gravitational wave using our model. From (22) we obtain a momentum input to the center of mass of

$$dP_z/dt = h_0 \cos(\omega_0 t + \varphi) \cos 2\omega t \frac{\omega^2 \omega_0 l^2 M}{24c} \tag{29}$$

where l is the length of the bar, M its mass, and h_0, ω_0, and φ are the amplitude, frequency and phase difference of the gravitational wave, respectively. This is for the case where the wave vector is perpendicular to the plane of rotation. In the ideal case $\omega \approx \omega_0/2, \varphi = 0$ or π, the mean energy input is given by

$$\dot{E} = \pm h_0 \frac{\omega_0^3 l^2 M}{192}. \tag{30}$$

Note that the change can be of either sign, thus the gravitational wave cannot only be absorbed, but can also stimulate the emission of gravitational waves from the system. For reasonable values (bar of $1m$, $1-100kg$ mass, $\omega/2\pi \sim 10-1000$ Hertz, $h_0 \sim 10^{-20}$) the attainable energy input ranges in about

$$\dot{E} \sim 10^{-13} ... 10^{-9} W. \tag{31}$$

Though this is quite large compared to the resonant detector, it seems questionable whether this change in the rotational energy of the bar can be measured. Certainly the acceleration of the rotating bar in the direction of the wave, given by (29), is too small to be detectable.

For a non-rotating mass, the lowest order contribution stems from the time derivatives of the positions and momenta induced by the wave and thus leads to a response of order h^2. This is the Weber detector. Here we have to calculate the forces on the constituent masses of the detector and solve the according equations of motion before we can estimate the response. As we indicated above, there exists a crucial difference between electromagnetically and gravitationally coupled systems.

7 Force distribution in a massive body

We first consider electromagnetically coupled bodies, as ion crystals and metals, where the gravitational forces between the constituents play no role. Ion crystals have the advantage that all charges can be considered to be pointlike and located on lattice points. Thus only discrete sums have to be evaluated.

7.1 Ion crystal

We first consider a gravitational wave propagating in z-direction incident on a lattice of ions. In order for the system to possess a stable ground state, we have to include not only the Coulomb potential, but also some (short-ranged) repulsive potential. The Coulomb force exerted by particle $s\prime$ on particle s is given, to first order in h, by

$$F^a_{C,\,ss\prime} = \frac{q_s q_{s\prime}}{4\pi\,\varepsilon_0} \left(\frac{x^a_{ss\prime}}{r^3_{ss\prime}} - h_{ab} \frac{x^b_{ss\prime}}{r^3_{ss\prime}} + \frac{3}{2} h_{bc} \frac{x^b_{ss\prime} x^c_{ss\prime} x^a_{ss\prime}}{r^5_{ss\prime}} \right), \tag{32}$$

for the x- and y- directions, the force in z-direction additionally involves the derivative of h, which is not of interest here. In the case of the Born-Meyer potential the repulsive potential is of exponential form

$$V_{BM}(r_{ss\prime}) = A_{ss\prime} e^{-\beta r_{ss\prime}} \tag{33}$$

with couplings $A_{ss\prime}$ that depend on the charges. Though it is necessary to include this potential, its precise form is not crucial for our further considerations. We obtain an additional force from this potential, up to first order in h given by

$$\begin{aligned} F^a_{BM,ss\prime} &= -\frac{\partial V_{BM},}{\partial x^{a\ (s)}}, \\ &= A_{ss\prime}\beta \frac{e^{-\beta r_{ss\prime}}}{r_{ss\prime}}\, x^a_{ss\prime} + \frac{1}{2}\, A_{ss\prime}\beta\, \frac{e^{-\beta r_{ss\prime}}}{r_{ss\prime}} \left(\frac{1}{r^2_{ss\prime}} + \beta \frac{1}{r_{ss\prime}} \right) h_{bc}\, x^b_{ss\prime}\, x^c_{ss\prime}\, x^a_{ss\prime} \\ &\quad -A_{ss\prime}\beta\, \frac{e^{-\beta r_{ss\prime}}}{r_{ss\prime}}\, h_{ab}\, x^b_{ss\prime}. \end{aligned} \tag{34}$$

The total force on ion s is given by

$$\begin{aligned}F_s^a &= \sum_{s'\neq s}\left(\frac{q_s q_{s'}}{4\pi\varepsilon_0}\,\frac{1}{r_{ss'}^3}+A_{ss'}\beta\frac{e^{-\beta r_{ss'}}}{r_{ss'}}\right)\,x_{ss'}^a \\ &-h_{ab}\sum_{s'\neq s}\left(\frac{q_s q_{s'}}{4\pi\varepsilon_0}\,\frac{1}{r_{ss'}^3}+A_{ss'}\beta\frac{e^{-\beta r_{ss'}}}{r_{ss'}}\right)\,x_{ss'}^b \\ &+h_{bc}\sum_{s'\neq s}\left(\frac{3q_s q_{s'}}{8\pi\varepsilon_0}\,\frac{1}{r_{ss'}^5}+\frac{1}{2}A_{ss'}\beta\frac{e^{-\beta r_{ss'}}}{r_{ss'}}\left(\frac{1}{r_{ss'}^2}+\frac{\beta}{r_{ss'}}\right)\right)\,x_{ss'}^b\,x_{ss'}^c\,x_{ss'}^a\end{aligned} \tag{35}$$

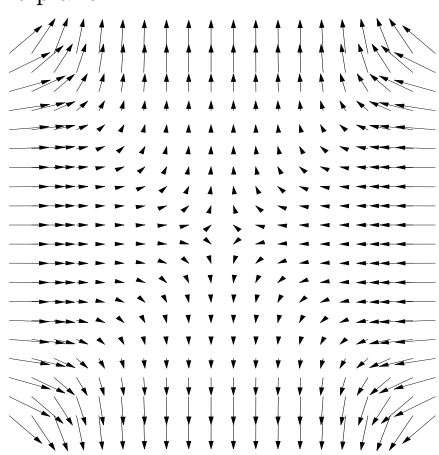
Assuming that the ion chain is in its unperturbed equilibrium state, both the first and second terms vanish because the vanishing of the first term defines the equilibrium in absence of a gravitational wave, and the second term is just the first multiplied by the matrix h_{ab}. Thus the gravitational wave gives rise to a perturbation of the equilibrium state induce by the force

$$\Delta F_s^a = \left[h_{bc}\sum_{s'\neq s}\left(\frac{3q_s q_{s'}}{8\pi\varepsilon_0}\,\frac{1}{r_{ss'}^5}+\frac{1}{2}A_{ss'}\beta\frac{e^{-\beta r_{ss'}}}{r_{ss'}}\left(\frac{1}{r_{ss'}^2}+\frac{\beta}{r_{ss'}}\right)\right)\,x_{ss'}^b\,x_{ss'}^c\right]\,x_{ss'}^a \tag{36}$$

Once the system is driven out of the equilibrium state, we still can ignore the second term in (35), because for deviations $\Delta x_{ss'}^a \sim h$ from equilibrium this term is only of order h^2. The first term then describes phonon-graviton interaction.

The sum over all other ion in (36) now depends on the dimension of the lattice. The sum over the short-ranged part converges even for an infinite lattice, so that its contribution is always limited. The sum over the Coulomb part is an alternating sum. In one dimension, this sum is always bounded by the first term that is not canceled by a contribution from a symmetric neighbor, thus the force has a maximum on the endpoints and decreases proportional to R^{-2} with the distance R from the endpoints. In two dimensions we find that the force distribution, in Fig. 1 plotted for a +-polarized wave, exhibit the correct quadrupole structure. Therefore acoustic modes are excited. This result is obtained only when the short-range potential is inculded. The Coulomb forces alone give rise to forces that also alternate in direction form ion to ion, so that we would have arrived at the wrong conclusion that optical modes are excited.

Figure 1: Force distribution (in arbitrary units) in a two-dimensional ion lattice as induced by a gravitational wave propagating perpendicular to the plane.



The forces decay rapidly away from the boundary and do not follow the linear law expected from the oscillator model. Thus a broad spectrum of modes is excited, instead of a single mode resonantly driven. We now present a general argument that this pertains to the relevant case of three dimensions.

7.2 General case

Since the temperature of the body must be low in order to achieve the desired sensitivity of a detector, we assume that the mass is a perfect crystal, either an ion crystal with discrete charges locate on some lattice, or a metal with ions on some lattice and the electron gas in between. The body is decomposed into a finite number of elementary cells V_i with their charge centers at $\vec{r}_i$, that are (i) electrically neutral,

$$\int_{V_i} d^D r \rho(\vec{r}) = 0 \tag{37}$$

and (ii) do not possess an electric dipole moment,

$$\int_{V_i} d^D r \, (\vec{r} - \vec{r}_i) \, \rho(\vec{r}) = 0. \tag{38}$$

D is the spatial dimension of the lattice. We now consider a charge element $q' = \rho(\vec{r}')dV$ located at $\vec{r}'$ and the perturbational force exerted on it by the elementary cell V located at $\vec{r}_i = 0$, according to (36). For $|\vec{r}'| \gg 1/\beta$ we can ignore the short-range potential and have

$$F_s^a(\vec{r}') = q' \int_V d^D r \, \frac{3\rho(\vec{r})}{8\pi\varepsilon_0} \frac{(\vec{r}' - \vec{r})^a}{|\vec{r}' - \vec{r}|^5} h_{bc} \, (\vec{r}' - \vec{r})^b \, (\vec{r}' - \vec{r})^c \,. \tag{39}$$

We expand the integrand into powers of $\vec{r} = (\ x, y, z)$ up to second order. Then the integrals of the first and second order vanish due to conditions (37) and (38), respectively. As an example, if h is of +-polarization, the forces in two dimensions are given by

$$F^x_{q',V}(\vec{r}') = q' h_+ \left[\frac{x'\left(6x'^2 - 9y'^2\right)}{2r'^7} I_1 + \frac{x'\left(-3x'^2 + 12y'^2\right)}{2r'^7} I_2 + \frac{y'\left(12x'^2 - 3y'^2\right)}{r'^7} I_3 \right] \tag{40}$$

$$F^y_{q',V}(\vec{r}') = q' h_+ \left[\frac{y'\left(12x'^2 - 3y'^2\right)}{2r'^7} I_1 + \frac{y'\left(-9x'^2 + 6y'^2\right)}{2r'^7} I_2 + \frac{x'\left(-3x'^2 + 12y'^2\right)}{r'^7} I_3 \right] \tag{41}$$

where the integrals

$$I_1 = \int_V d^D r \, x^2 \, \frac{3\rho(\vec{r})}{8\pi\varepsilon_0}, \quad I_2 = \int_V d^D r \, y^2 \, \frac{3\rho(\vec{r})}{8\pi\varepsilon_0}, \quad I_3 = \int_V d^D r \, xy \, \frac{3\rho(\vec{r})}{8\pi\varepsilon_0} \tag{42}$$

describe the quadrupole moments of the electric charge distribution in the elementary cell. The generalization to three dimensions is straightforward, with the same qualitative properties: The force exerted by some elementary cell on a charge element q' always decreases at least proportional to r'^{-4} with the distance between the charge and the cell. If the quadrupole moments vanish, as for cubic lattices, the decrease is even faster by two powers of r'.This implies that for a given charge element q' the sum over all elementary cells always converges in $D = 1, 2$, or 3 dimensions. Therefore the force on any ion in the body is bounded independently from the size of the body,

$$\left|\vec{F}_{q'}\right|_{\max} \leq \sum_i \left|F^y_{q',V_i}(\vec{r}' - \vec{r}_i)\right| \leq const \sum_i \frac{1}{|\vec{r}' - \vec{r}_i|^4}. \tag{43}$$

Further, the reflection symmetry of the lattice implies that all forces from cells in a volume that possesses reflection symmetry around the charge element add up to zero. Hence also in the general case the charge element feels a force that depends on its distance R to the surface of the body. A crude estimate gives

$$\left|\vec{F}_{q'}\right|_R \leq \sum_{i, |\vec{r}' - \vec{r}_i| > R} \left|F^y_{q',V_i}(\vec{r}' - \vec{r}_i)\right| \leq const \sum_{|\vec{r}' - \vec{r}_i| > R} \frac{1}{|\vec{r}' - \vec{r}_i|^4} \sim R^{D-4}. \tag{44}$$

Depending on the dimension, we observe that the force decreases with the distance from the surface, at least with $1/R$ in three dimensions, if the quadrupole moments I_1, I_2, I_3 all vanish the decrease is even stronger. Scaling with the lattice constant a leads us to

$$\left|\vec{F}_{q'}\right|_R \leq \left|\vec{F}_{q'}\right|_{\max} \left(\frac{a}{R}\right)^{4-D}. \tag{45}$$

Because the sum over the forces from the short range potential decreases even faster, the total perturbational forces, which are maximal on the surface, decrease to about 10^{-3} of the surface value within 1000 atomic layer, which is about 1micrometer. When we integrate the forces (40,41) over an elementary cell in order to obtain the mean force on the cell, we again loose two powers due to (37) and (38), resulting in mean forces between two cells V_i and V_j that decrease at best proportional to $|\vec{r}_j - \vec{r}_i|^{-6}$. Thus we conclude that the bulk of the material remains unaffected by the gravitational wave. This result has its origin in the nature of the electromagnetic coupling with its charges of different signs.

7.3 Gravitationally coupled matter

Clearly, the result of the preceding section was due to the conditions (37,38), but (37) does not hold for the attractive gravitational forces. The acceleration of a test mass is given by

$$\vec{a}_G\left(\vec{r}'\right) = \int_V d^D r \; \frac{\rho_m(\vec{r})}{G} \; \frac{(\vec{r}' - \vec{r})}{\left|\vec{r}' - \vec{r}\right|^5} h_{bc} \; \left(\vec{r}' - \vec{r}\right)^b \left(\vec{r}' - \vec{r}\right)^c, \tag{46}$$

where ρ_m is the mass density of the body. Assuming that the wavelength of the gravitational wave is large compared to the dimensions of a homogeneous massive body, we can take ρ_m and h to be constant and are able to evaluate the integral for simple geometries. For a +-polarized wave propagating in z-direction the maximal acceleration on the surface is calculated to

$$|\vec{a}_G|_{\max} = \begin{cases} \rho_m \pi r \left(1 - \frac{1}{\sqrt{1+\frac{l^2}{r^2}}}\right) \text{ for a cylinder of radius } r \text{ and lenght } l \\ \rho_m \pi \frac{\tan^2\phi}{(1+\tan^2\phi)^{3/2}} l \text{ for the tip of a cone of opening angle } \phi \text{ and height } l \\ \rho_m \pi \frac{8R}{15} \text{ at the surface a sphere of radius } R. \end{cases} \tag{47}$$

Naturally, there exists an angular dependence of the forces of quadrupole characteristic. Thus the forces grow linearly with the linear dimensions of the massive body. The maximal acceleration on the equator of a sphere thus is given in the TT frame of reference by

$$|\vec{a}_G|_{\max}^{TT} = \frac{2}{5} |h| \, g \tag{48}$$

where g is the surface acceleration of the mass. This force will truly excite the quadrupole modes of a body and is able to do work against the gravitational and electromagnetic forces that keep the body together. Comparing this with the apparent acceleration in the PFR system,

$$|\vec{a}_G|_{\max}^{PFR} = \frac{1}{2} \omega^2 R \, |h| \,, \tag{49}$$

we see that (48) is several orders of magnitude smaller that (49). In the PFR system the forces are primarily virtual forces introduced by a time-dependent coordinate transformation. The influence of the gravitational wave field on the fundamental interactions, gravitational as well as electromagnetic ones, must be taken into account. Under the motion of the forces (49) the massive body remains practically in its ground state, and is not exited, because these forces do not do work against the other forces that also follow the gravitational wave field. In the common picture of a spring this means that the equilibrium length of the spring just follows the motion of the particles in such a way that only the residual forces (48) and (39) can do work.

8 New Type of Detector?

Though we have shown that the metric effects on the electromagnetic coupling cancel in the bulk of a massive body, the question remains whether the surface effect we found could be measurable. Similar to expression (48) for the gravitationally coupled matter, we can estimated the acceleration induce by the gravitational wave on the surface of an ion lattice basically by

$$|\vec{a}_{EM}|_{\max}^{TT} \sim \left| \frac{Z_+ Z_-}{m_\pm d^2 \pi \varepsilon_0} h \right| \tag{50}$$

where Z_+, Z_- and m_+, m_- are the charges and the masses of the ions, respectively, and d the lattice constant. A geometric factor of order 1 has to be included. This factor will depend on the structure of the lattice and the precise behavior of the repulsive potential. The numerical value of (50) is of the order of $h \cdot 10^{16} ms^{-2}$ (for KCl) which is several orders of magnitude larger than (49). This raises the question whether a detector can be built on this basis, e.g. using the piezoelectric effect. We will study this possibility elsewhere.

9 Conclusions

We have analyzed the resonant mass gravitational wave detector form a microscopic point of view, using the wave guide (TT) frame of reference. In this system the variation of the Coulomb field of the constituent charges of the body give the dominant contribution. We expressed the response to a metric perturbation through a microscopic response function of quadrupole nature. We identified the two basic detector types, the rotational Braginsky detector and the resonant Weber detector. To our great surprise, we could not validate the results for the Weber detector that are obtained when the system is modeled by a resonant spring. For the electromagnetically coupled body, we showed that the force distribution is such that the forces are restricted to a small surface layer, with not bulk force. For the gravitationally coupled body, we observe a linear force law, but orders of magnitude smaller than expected from the calculations in the preferred frame of reference (PFR). We attributed the reason for the failure of the spring model to the negligence of the influence of the wave on the electromagnetic and gravitational fields in this model. In the PFR the forces therefore are primarily of virtual nature, not doing work against the other forces. We conclude that the excitation of resonance of the fundamental quadrupole modes of a massive body by gravitational waves is by far smaller than currently expected, possibly too small for detection.

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